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Note

Borel-Type Sums Using Two-Point Rational Approximants

In a recent paper Wimp [1] considers the computations of Borel-type sums

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} s_n , \tag{1}$$

where $\{s_n\}$ is some fixed convergent sequence. For small values of x, (1) can be used to evaluate $f(x)$, while for x large it was shown how the function could be evaluated from its asymptotic expansion. The asymptotic expansion was found, by Wimp, using the Borel transform.

The purpose of this note is to indicate how a rational approximation to $f(x)$ can be constructed from (1) and its asymptotic expansion. These approximations, here written in the form of continued fractions, are usually termed two-point rational approximants. They have the advantage that they usually converge quicker than either of the associated series; second it is more convenient to use a single approximating function in $[0, \infty]$ rather than two which do not have an overlap domain of convergence.

For completeness we give a brief account of the construction and use of two-point rational approximants via continued fractions. Suppose a function $f(z)$, z may be complex, can be expanded about $z = 0$ in the series

$$f(z) = \sum_{n=0}^{N} c_n z^n + \epsilon_N(z) \tag{2}$$

and about $z = \infty$ in the series

$$f(z) = \sum_{n=1}^{M} b_n z^{-n} + \delta_M(z). \tag{3}$$

The series (2) and (3) may be convergent or asymptotic and $\epsilon_N(z)$ and $\delta_M(z)$ are the respective remainders. Here neither c_0 nor b_1 are assumed to be zero. Rational fractions, in the form of continued fractions, are now constructed which will fit $m + 2r = p < N$ terms of (2) and $m = q < M$ terms of (3).

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These are of the form, McCabe and Murphy [2],

$$M_{p,q} = \frac{n_1}{(1 + d_1 z)} + \frac{n_2 z}{(1 + d_2 z)} + \cdots + \frac{n_m z}{(1 + d_m z)}$$
$$+ \frac{n_{m+1} z}{1} + \frac{n_{m+2} z}{1} + \cdots + \frac{n_{m+2r-1} z}{1} + \frac{n_{m+2r} z}{1}. \tag{4}$$

The coefficients $n_1, n_2, \ldots, n_m$; $d_1, \ldots, d_m$ can be calculated in a number of ways but probably the best is that due to Viscovatoff [3] (see Khovanskii [4]) with modifications due to O'Donohoe [5], see also [6], using the corresponding sequence (CS) algorithm. The remaining coefficients are calculated via determinants McCabe and Murphy [2]. Clearly (4) is simply a rational fraction and can be found directly in this form. This approach was followed by Frost and Harper [7]; although this is more direct it has the drawback, with its implications, that the coefficients of the rational fractions depend on p and q, while the n's and d's in (4) are independent of these parameters.

Let us now apply these results to (1). It was shown by Wimp that if

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n s_n}{n!}$$

with

$$s_n \sim s + \lambda^n \left[\frac{C_1}{n} + \frac{C_2}{n^2} + \cdots\right], \qquad n \to \infty$$

and

$$|\lambda| \leqslant 1, \qquad \lambda \neq 0$$

then

$$f(x) \sim s + e^{x(\lambda-1)} \left[\frac{\bar{C}_2}{x} + \frac{\bar{C}_2}{x^2} + \cdots\right], \qquad \text{as} \quad x \to \infty \tag{5}$$

where

$$\bar{C}_s = \lambda^{-s} \sum_{k=1}^{s} A_{k,s} C_k,$$

with $A_{k,s}$ as defined in [1]. If we now construct

$$g(x) = \{f(x) - s\}\, e^{-x(\lambda-1)}, \tag{6}$$

then $g(x)$ has the series developments of the form (2) and (3) about $x = 0$ and ∞. The rational approximants $M_{p,q}(x)$ to $g(x)$ can now be constructed to give finally

$$f(x) = M_{p,q}(x)\, e^{x(\lambda-1)} + s. \tag{7}$$

As an illustration consider the first example of Wimp [1]. There

$$\phi(\alpha, \beta) = e^{-\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!\, n^2} \left\{1 + \frac{\beta}{n^2}\right\}^{-3/2} \tag{8}$$

with α and β as in [1]. Here

$$s_n = \frac{\{1 + \beta/n^2\}^{-3/2}}{n^2},$$

$\lambda = 1$, and $s = 0$. Due to the behavior of ϕ at $\alpha = 0$ and ∞ we construct the rational approximant $M_{p,q}(\alpha, \beta)$ to

$$\alpha^{-2}\{\phi(\alpha, \beta) + \alpha(1 + \beta)^{-3/2}\}$$

and hence

$$\phi(\alpha, \beta) = \alpha^2 M_{p,q}(\alpha, \beta) + \alpha(1 + \beta)^{-3/2}. \tag{9}$$

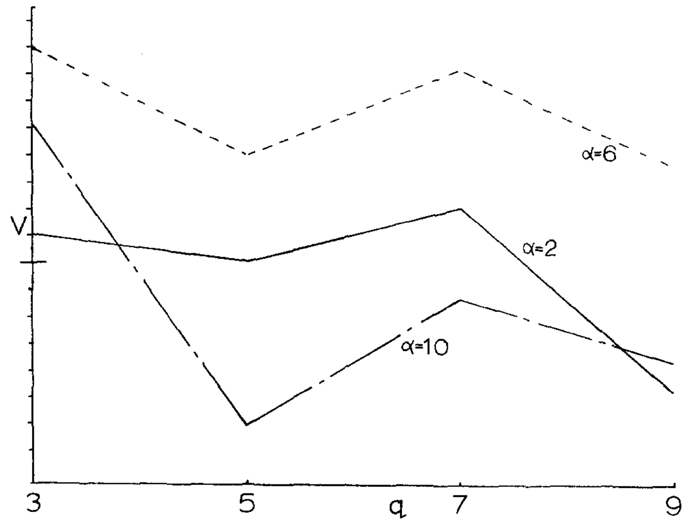

FIGURE 1

In Fig. 1 we plot the relative errors of $M_{p,q}(\alpha, \beta)$ in (9) for $\beta = 1.0$ and $\alpha = 2, 6, 10$, with $p = 9$ and $q = 3, 5, 7, 9$. The scale mark for V is at 50 for $\alpha = 2$, 55 for $\alpha = 6$, and 15 for $\alpha = 10$, the scale intervals are unity. Also

$$\begin{aligned} M_{p,q}(\alpha, \beta) &= -0.13473900 + V \times 10^{-8} \quad \text{when} \quad \alpha = 2, \\ M_{p,q}(\alpha, \beta) &= -0.0576 + V \times 10^{-6} \quad \text{when} \quad \alpha = 6, \\ M_{p,q}(\alpha, \beta) &= -0.0532 + V \times 10^{-6} \quad \text{when} \quad \alpha = 10. \end{aligned}$$

For $\alpha = 10$, the fitting of more terms of the asymptotic expansion, i.e., increasing q, results in a convergent sequence for $M_{p,q}(\alpha)$. The relative error decreases but the absolute error is of $O(10^{-6})$ for $q = 9$ and $p = 9$. For $\alpha = 2$, the relative error increases with q, indicating that the asymptotic expansion is having a detrimental affect on the convergence. The absolute error here for $p = 9$, $q = 9$ is $O(10^{-8})$. For intermediate values of α the situation is, as one would expect, a combination of the extreme situations. For instance for $\alpha = 6$, the relative error as q increases is almost constant, while the absolute error is $O(10^{-6})$. Similar considerations apply to the other example quoted by Wimp.

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